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TWO-PHASE FLOW MODELLING: THE CLOSURE ISSUE FOR A TWO-LAYER FLOW

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Abstract—The fully-developed, laminar flow of two fluid layers in a horizontal channel is studied by means of the height-averaged balance equations. The closure issue is addressed and the closure relations for the wall and interfacial shear stresses are given for some particular cases.

Key Words: two-phase flow modelling, averaged equations, closure relations

1. INTRODUCTION

Consider the two-dimensional, steady-state, fully-developed, laminar flow of two fluid layers in a horizontal channel without interfacial phase change. The fluids are incompressible and their viscosities are constant. Given the volumetric flow rates Q_1 and Q_2 per unit width of the channel, let us ask ourselves how to determine the height fractions ϵ_1 and ϵ_2 (figure 1) and the pressure gradient by using a two-fluid model based upon balance equations that are averaged over the height of each fluid layer. Actually, the closure issue can be stated as follows: what are the relations that we must choose to express (a) the topological law, (b) the interfacial friction, (c) the wall friction on fluid 1 and (d) the wall friction on fluid 2, in order to obtain the same height fractions ϵ_1 and ϵ_2 and the same pressure gradient as those calculated by means of the local instantaneous equations? This problem appears trivial but is surely not.

2. THE LOCAL INSTANTANEOUS PROBLEM

The solution is obtained from the following set of equations: mass balance equations for fluids 1 and 2; momentum balance equations for fluids 1 and 2 and for the interface, no-slip conditions at the interface and on the walls. The height fractions are given by the following equation:

$$Q_1\left(\frac{4-\epsilon_1}{\epsilon_1}+\frac{\mu_1}{\mu_2}\frac{\epsilon_2^2}{\epsilon_1^2}\right)=Q_2\left(\frac{4-\epsilon_2}{\epsilon_2}+\frac{\mu_2}{\mu_1}\frac{\epsilon_1^2}{\epsilon_2^2}\right), \qquad [1]$$

where μ_1 and μ_2 are the viscosities of the fluids and

$$\epsilon_1+\epsilon_2=1. \qquad [2]$$

Equation [1] is a fourth-order polynomial in ϵ_1 or ϵ_2 that has already been found by Bowen (1973). The pressure gradient G is given by

$$G=-\frac{3}{H^3}\left(\mu_1\frac{Q_1}{\epsilon_1^2}+\mu_2\frac{Q_2}{\epsilon_2^2}\right), \qquad [3]$$

where H is the channel height. It should be noted that [1]–[3] show that the triangular relationship suggested by Hewitt & Hall-Taylor (1970) does hold, contrary to what had been claimed by Bowen (1973). Actually, given the flow rate of one of the fluids, the height of the same fluid can be calculated if the pressure gradient is known.

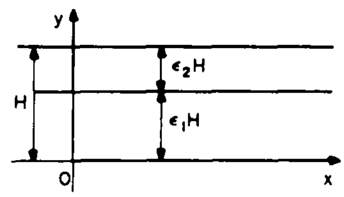

Figure 1. The steady-state flow of two fluid layers.

3. THE SAME PROBLEM SOLVED WITH THE HEIGHT-AVERAGED EQUATIONS

The following equations should be used:

mass balance for fluid 1,

$$\frac{\mathrm{d}}{\mathrm{d}x} \epsilon_1 H \langle u_1 \rangle = 0; \tag{4}$$

mass balance for fluid 2,

$$\frac{\mathrm{d}}{\mathrm{d}x} \epsilon_2 H \langle u_2 \rangle = 0; \tag{5}$$

mass balance at the interface, this equation is identically satisfied due to the absence of phase change;

momentum balance for fluid 1,

$$\epsilon_1 H \frac{\mathrm{d}\langle p_1 \rangle}{\mathrm{d}x} = -\tau_{1i} - \tau_{1w}; \tag{6}$$

momentum balance for fluid 2,

$$\epsilon_2 H \frac{\mathrm{d}\langle p_2 \rangle}{\mathrm{d}x} = -\tau_{2i} - \tau_{2w}; \tag{7}$$

momentum balance at the interface,

$$p_{1i} = p_{2i} \triangleq p_i \tag{8}$$

and

$$\tau_{1i} = -\tau_{2i} \triangleq \tau_i. \tag{9}$$

Given H we have four differential equations, [4]–[7], five dependent variables, namely ϵ_1, $\langle u_1 \rangle$, $\langle u_2 \rangle$, $\langle p_1 \rangle$, $\langle p_2 \rangle$, and three supplementary variables, namely τ_i, τ_{1w} and τ_{2w}; p is the pressure and τ is the shear stress.

Considering [4]–[7] without looking at figure 1, it is impossible to say what are the respective positions of fluids 1 and 2. This is a naive but important remark. If fluid 1, which is supposed to be the denser one, is above or below fluid 2 the flow is unstable or stable. This feature surely cannot be predicted by [4]–[7]. Consequently, as we need a supplementary equation to eliminate one of the dependent variables, this equation must contain some information regarding the respective position of the fluids. The transversal momentum equations of the local instantaneous problem lead immediately to the following result:

$$\begin{cases} \langle p_1 \rangle = p_i + \frac{1}{2}\rho_1 g \epsilon_1 H & [10] \\ \langle p_2 \rangle = p_i - \frac{1}{2}\rho_2 g \epsilon_2 H. & [11] \end{cases}$$

Thus, we can say that we have the following topological relation:

$$\langle p_2 \rangle = \langle p_1 \rangle - \tfrac{1}{2}(\epsilon_1 \rho_1 + \epsilon_2 \rho_2) g H. \tag{12}$$

This concept of a topological equation was discussed in detail by Delhaye (1988). If we consider a flow which is different from the one defined in figure 1, for instance a bubbly flow or a churn flow, the equations analogous to [10] and [11] cannot be obtained directly from the local instantaneous problem. They should result from physical intuition and an analysis specific to each case.

As a result we have:

$$\frac{\mathrm{d}\langle p_1\rangle}{\mathrm{d}x} = \frac{\mathrm{d}\langle p_2\rangle}{\mathrm{d}x} \triangleq G. \qquad [13]$$

From the momentum equations we thus obtain

$$G = -\frac{1}{H}(\tau_{1\mathrm{w}} + \tau_{2\mathrm{w}}) \qquad [14]$$

and

$$\epsilon_1 = \frac{\tau_\mathrm{i} + \tau_{1\mathrm{w}}}{\tau_{1\mathrm{w}} + \tau_{2\mathrm{w}}}. \qquad [15]$$

The closure issue can now be stated as follows: find three relations for τ_i, $\tau_{1\mathrm{w}}$ and $\tau_{2\mathrm{w}}$ so that ϵ_1 and G calculated from [14] and [15], respectively, are the same as ϵ_1 and G calculated from [1] and [3], respectively.

4. GENERAL SOLUTION OF THE LOCAL INSTANTANEOUS PROBLEM

Let us define

$$q \triangleq \frac{Q_1}{Q_2} \qquad [16]$$

and

$$m \triangleq \frac{\mu_1}{\mu_2}, \qquad [17]$$

Choosing q and m positive, it is possible to show that [1] has one and only one root ϵ_1 lying between 0 and 1. This root can be obtained exactly but its expression is very complicated. However, q is easily obtained as a function of ϵ_1 and m (figure 2). Defining

$$X \triangleq \frac{\epsilon_1}{\epsilon_2}, \qquad [18]$$

equation [1] reads

$$X^4 + 4mX^3 + 3mX^2(1-q) - 4qmX - qm^2 = 0. \qquad [19]$$

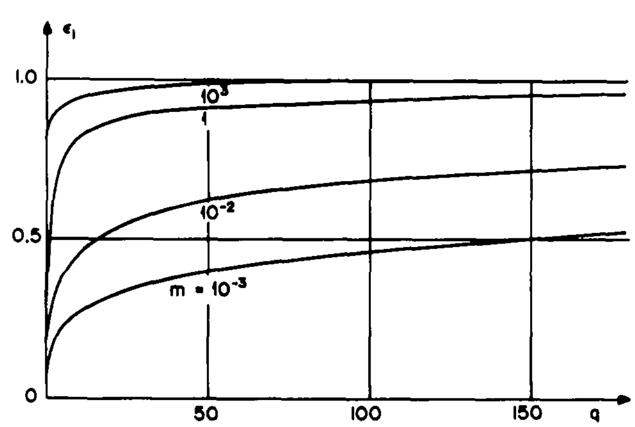

Figure 2. The solution of [1].

5. A PARTICULAR SOLUTION OF THE LOCAL INSTANTANEOUS PROBLEM

When the viscosities of the two fluids are equal ($m = 1$), [19] becomes

$$(X+1)(X^3+3X^2-3qX-q)=0. \quad [20]$$

The only physical solution of [20] reads

$$X = 2\sqrt{q+1}\cos\theta - 1, \quad [21]$$

where

$$\theta \triangleq \tfrac{1}{3}\arccos\left(-\frac{1}{\sqrt{q+1}}\right). \quad [22]$$

As a result, we have

$$\epsilon_1 = \frac{X}{1+X} = \frac{2\sqrt{q+1}\cos\theta - 1}{2\sqrt{q+1}\cos\theta}. \quad [23]$$

6. ASYMPTOTIC SOLUTIONS OF THE LOCAL INSTANTANEOUS PROBLEM

(i) Case where $m \to 0$

All the roots of [19] tend to zero. As [19] is a fourth-order polynomial, $X \to 0$ as a power of m. One finds

$$X \simeq [4qm + 3(q-1)m(4qm)^{1/3} + 3m^2(4qm)^{-1/3}(q^2-7q+1)]^{1/3}. \quad [24]$$

(ii) Case when $q \to 0$

One finds

$$X \simeq \tfrac{1}{3}qm + \tfrac{4}{3}q(\tfrac{1}{3}qm)^{1/2}(1-\tfrac{1}{3}m) + \tfrac{8}{27}mq^2. \quad [25]$$

7. THE SHEAR STRESSES AS OBTAINED FROM THE LOCAL INSTANTANEOUS EQUATIONS

The determination of the velocity profiles enables the wall and interfacial shear stresses to be calculated:

$$\tau_{1w} = +\frac{\epsilon_1}{\epsilon_2^2}\mu_2\frac{Q_2}{H^2} + \frac{\epsilon_1+2}{\epsilon_1^2}\mu_1\frac{Q_1}{H^2}, \quad [26]$$

$$\tau_{2w} = +\frac{\epsilon_2}{\epsilon_1^2}\mu_1\frac{Q_1}{H^2} - \frac{\epsilon_1-3}{\epsilon_2^2}\mu_2\frac{Q_2}{H^2}, \quad [27]$$

and

$$\tau_i = -2\frac{\epsilon_2}{\epsilon_1^2}\mu_1\frac{Q_1}{H^2} + 2\frac{\epsilon_1}{\epsilon_2^2}\mu_2\frac{Q_2}{H^2}. \quad [28]$$

The temptation would be to claim that [26]–[28] are the closure relations that are looked for to solve the problem with the height-averaged equations. However, by replacing the wall and interfacial shear stresses in [15] by their expressions [26]–[28] would not lead to any new additional relation. On the contrary, they obviously lead to a trivial identity. Actually the closure relations for the wall and interfacial shear stresses are obtained by replacing ϵ_1 and ϵ_2 as functions of m and q (from the solution of [19] or from [23]) in [26]–[28]. The general solutions can be obtained after lengthy calculations. Particular and asymptotic solutions are easier to handle and will be dealt with in the next paragraphs. In the following, these solutions will be called closure relations in accordance with the general case where the local instantaneous problem is of no help because of its complexity.

8. PARTICULAR SOLUTIONS FOR THE SHEAR STRESS CLOSURE RELATIONS

When the viscosities of the two fluids are equal, the shear stresses are given by

$$\tau_{1w} \equiv \tau_{2w} = 6\mu \frac{Q_1 + Q_2}{H^2} \qquad [29]$$

and

$$\tau_i = 6\mu \frac{Q_1 + Q_2}{H^2}\left(\frac{\sqrt{q+1}\cos\theta - 1}{\sqrt{q+1}\cos\theta}\right). \qquad [30]$$

9. ASYMPTOTIC SOLUTIONS FOR THE SHEAR STRESS CLOSURE RELATIONS

(i) Case where $m \to 0$

One obtains to the first order

$$\tau_{1w} \equiv \tau_i \simeq 3\frac{\mu_2 Q_2}{H^2}\left(\tfrac{1}{2}qm\right)^{1/3} \qquad [31]$$

and

$$\tau_{2w} \simeq 3\frac{\mu_2 Q_2}{H^2}. \qquad [32]$$

(ii) Case where $q \to 0$

One obtains to the first order

$$\tau_{1w} \equiv \tau_{2w} \equiv -\tau_i \simeq 6\frac{\mu_2 Q_2}{H^2}. \qquad [33]$$

10. A RELATION BETWEEN THE WALL AND INTERFACIAL SHEAR STRESSES FOR THE PROBLEM CONSIDERED IN FIGURE 1

In each layer, the local instantaneous momentum balance reads

$$-\frac{dp}{dx} + \mu\frac{d^2u}{dy^2} = 0. \qquad [34]$$

Multiplying by $2\,du/dy$, we obtain

$$-2\frac{dp}{dx}\frac{du}{dy} + 2\mu\frac{du}{dy}\frac{d^2u}{dy^2} = 0. \qquad [35]$$

Integrating over the heights occupied by the two fluids we get

$$-2\frac{dp}{dx}u_{1i} + \mu_1\left(\frac{du_1}{dy}\right)_i^2 - \mu_1\left(\frac{du_1}{dy}\right)_{1w}^2 + 2\frac{dp}{dx}u_{2i} + \mu_2\left(\frac{du_2}{dy}\right)_{2w}^2 - \mu_2\left(\frac{du_2}{dy}\right)_i^2 = 0. \qquad [36]$$

On the interface one has

$$u_{1i} \equiv u_{2i}, \qquad [37]$$

thus one obtains

$$\mu_2\left[\left(\frac{du_2}{dy}\right)_{2w}^2 - \left(\frac{du_2}{dy}\right)_i^2\right] = \mu_1\left[\left(\frac{du_1}{dy}\right)_{1w}^2 - \left(\frac{du_1}{dy}\right)_i^2\right]. \qquad [38]$$

As we have

$$\tau_{1w} \triangleq \mu_1\left(\frac{du_1}{dy}\right)_{1w}, \quad \tau_{2w} \triangleq -\mu_2\left(\frac{du_2}{dy}\right)_{2w} \qquad [39]$$

and

$$\tau_i \triangleq -\mu_1 \left(\frac{\mathrm{d}u_1}{\mathrm{d}y}\right)_i \equiv -\mu_2 \left(\frac{\mathrm{d}u_2}{\mathrm{d}y}\right)_i. \qquad [40]$$

Equation [38] can also be written in the following form:

$$\mu_1(\tau_{2w}^2 - \tau_i^2) = \mu_2(\tau_{1w}^2 - \tau_i^2). \qquad [41]$$

This relation must be considered as a compatibility condition between the wall and interfacial shear stresses which must be fulfilled by any closure relation. Such is the case for [26]–[28] when [1] is taken into account.

11. PHYSICAL SIGNIFICANCE OF [35]

The local instantaneous momentum balances [34] written for each layer are the Euler–Lagrange equations corresponding to the stationarity of the following variational principle:

$$I \triangleq \int_0^{\epsilon_1 H} \mathscr{L}_1 \,\mathrm{d}y + \int_{\epsilon_1 H}^{H} \mathscr{L}_2 \,\mathrm{d}y, \qquad [42]$$

where the Lagrangian $\mathscr{L}_k$ ($k = 1, 2$) is given by

$$\mathscr{L}_k \triangleq \frac{1}{2}\left(\frac{\mathrm{d}u_k}{\mathrm{d}y}\right)^2 + \frac{1}{\mu_k}\frac{\mathrm{d}p}{\mathrm{d}x}u_k. \qquad [43]$$

The Legendre transform of [43] gives the expressions for the Hamiltonian for each layer:

$$\mathscr{H}_k = \frac{1}{2}\left(\frac{\mathrm{d}u_k}{\mathrm{d}y}\right)^2 - \frac{1}{\mu_k}\frac{\mathrm{d}p}{\mathrm{d}x}u_k. \qquad [44]$$

Equation [35] allows us to conclude that, for the problem considered in figure 1, the Hamiltonian $\mathscr{H}_k$ is a constant over the cross section of layer k. As a result we can think of

$$\frac{1}{2}\left(\frac{\mathrm{d}u_k}{\mathrm{d}y}\right)^2 \text{ as being a generalized kinetic energy}$$

and

$$-\frac{1}{\mu_k}u_k\frac{\mathrm{d}p}{\mathrm{d}x} \text{ as being a generalized potential energy.}$$

The value of the Hamiltonian $\mathscr{H}_k$ can be easily obtained, and is given by

$$\mathscr{H}_k = \frac{1}{2\mu_k^2}\tau_{kw}^2. \qquad [45]$$

12. PHYSICAL SIGNIFICANCE OF [38]

Equation [38] can be written in the following form:

$$\Delta_{2w} - \Delta_{2i} = \Delta_{1w} - \Delta_{1i}, \qquad [46]$$

where Δ_{kw} and Δ_{ki} ($k = 1, 2$) are the entropy sources at the wall and at the interface, given by

$$\Delta_{kw} = \frac{1}{T}\mu_k\left(\frac{\mathrm{d}u_k}{\mathrm{d}y}\right)_w^2, \quad \Delta_{ki} = \frac{1}{T}\mu_k\left(\frac{\mathrm{d}u_k}{\mathrm{d}y}\right)_i^2; \qquad [47]$$

T being the temperature of the system.

It is important to note that the results obtained in sections 10–12 apply only to the problem considered in figure 1, i.e. a two-layer flow with a flat interface which is indeed a flow without inertia.

Table 1. The closure relations for the two-layer flow problem solved with the height-averaged equations

	General case	Equal viscosities	$m \to 0$	$q \to 0$
Topological equation	[12]	[12]	[12]	[12]
Interfacial friction		[30]	[31]	[33]
Wall shear stress on fluid 1	Without interest because of the complexity of the equations	[29]	[31]	[33]
Wall shear stress fluid 2		[29]	[32]	[33]

13. CONCLUSIONS

The closure relations for the problem considered, i.e. a two-layer flow with a flat interface, are given in table 1. These relations are quite complicated, even in the particular or asymptotic cases, and could not have been postulated *a priori.*

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